A Comment on the Degrees of Freedom in the Ashtekar Formulation for 2+1 Gravity

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ABSTRACT

We show that the recent claim that the 2+1 dimensional Ashtekar formulation for General Relativity has a finite number of physical degrees of freedom is not correct.

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In a recent paper [1] Manojlović and Miković claim that the number of degrees of freedom in Ashtekar's formulation for 2+1 dimensional General Relativity is finite at variance with previous results of the authors [2]. We stand by the results of that paper wherein we were able to prove that, in spite of having the same number of first class constraints as phase space variables per point, the number of degrees of freedom in this formulation is *infinite*. In this comment we show that [1] is incorrect on several counts.

(i) The statement appearing in page 3034:

> *"..., by performing a gauge transformation on a null connection, one can always reach a non-null connection ..."*

is not true. This statement is 'proved' by (25) of [1]. However, (25) of that paper is incorrect because the gauge transformations of the connection that the authors have used are wrong. Specifically, eq.(17) should say

$$\delta A_1 = -\frac{d\epsilon^1}{d\theta} + (E_2 f_3 - E_3 f_2)\epsilon^3$$

and eq.(19) should be

$$\delta A_3 = -A_2\epsilon^1 + f_3\epsilon^2 + E_1 f_2\epsilon^3$$

The origin of the error in the last equation can be traced back to the use of an incorrect symplectic structure to derive it from the constraints. $(E^a_I \dot{A}^I_a)$ evaluated on (9), (10) of [1] is $E_1\dot{A}_1 + E_2\dot{A}_2 - E_3\dot{A}_3$ *not* $E_1\dot{A}_1 + E_2\dot{A}_2 + E_3\dot{A}_3$ as in (14) of [1]. This is because I is an SO(2,1) index and $t_I t^I = -1$ *not* $+1$. Correction of these errors leads to the following version of equation (25)

$$\begin{aligned}\delta(f_2 \pm f_3) \quad = \quad & -\epsilon^1(f_3 \pm f_2) + [\epsilon^2(f_2 \pm f_3)]' - \epsilon^2 A_1(f_3 \pm f_2) \\ & +[\epsilon^3 E_1(f_3 \pm f_2)]' - \epsilon^3 A_1 E_1(f_2 \pm f_3) - \epsilon^3(E_2 f_3 - E_3 f_2)(A_3 \pm A_2),\end{aligned} \tag{1}$$

where

$$f_2 \equiv \frac{dA_2}{d\theta} - A_1 A_3, \qquad f_3 \equiv \frac{dA_3}{d\theta} - A_1 A_2.$$

¿From (1) it is straightforward to obtain

$$\begin{aligned}\delta(f_2^2 - f_3^2) &= 2(\epsilon^2)'(f_2^2 - f_3^2) + \epsilon^2(f_2^2 - f_3^2)' + \epsilon^3\Big[2A_1E_1(f_3^2 - f_2^2) + \\ &\quad 2E_1(f_2f_3' - f_3f_2') + 2(E_2f_3 - E_3f_2)(A_2f_3 - A_3f_2)\Big] \qquad (2)\end{aligned}$$

At an interior point in a null patch, we have $f_2^2 - f_3^2 = 0$, $(f_2^2 - f_3^2)' = 0$, and $f_2 = \pm f_3$ so, modulo the constraints, [1] (2) gives $\delta(f_2^2 - f_3^2) = 0$. Clearly, this shows that the statement on pg 3034 of [1], referred to above is incorrect.

(ii) The authors claim that the holonomies around loops in flat patches are not gauge invariant objects. This claim is not correct. Note that we did *not* specify the location of the loops in fixed coordinates but demanded that they be associated with flat patches (since the connection is flat the holonomy is independent of the location of the loop within a flat patch). Hence, the loops are moved by the same diffeomorphism which moves the flat patches under evolution. Thus these observables are 2d diffeomorphism invariant besides obviously being SO(2,1) gauge invariant. Since these are all the gauge generated by the constraints on the connection part of the patch data, the holonomies are gauge invariant observables. Indeed, it is easily seen that (denoting the holonomy around the loop γ in a flat patch by H_γ),

$$\{H_\gamma, \mathcal{D}_a E^a_I\} = 0 \qquad (3)$$

$$\{H_\gamma, E^a_I F^I_{ab}\} = \{H_\gamma, E^a_I\}F^I_{ab} = 0 \qquad (4)$$

$$\{H_\gamma, \epsilon^{IJK}E^a_I E^b_J F_{abK}\} = \{H_\gamma, \epsilon^{IJK}E^a_I E^b_J\}F_{abK} = 0. \qquad (5)$$

where we used $F^I_{ab} = 0$ in (4),(5). These Poisson bracket relations are *independent* of the degeneracy of the metric.

(iii) The authors have misinterpreted our display of initial data in [2] as a gauge fixing condition. We have not fixed any gauge in that work! We summarize the arguments of [2] to emphasize this point. We studied the action of the gauge transformations

[1] Note that for flat curvatures, $f_2 = f_3 = 0$ and the last term in (2) vanishes. For non flat, null curvatures $f_2 = \pm f_3 \neq 0$ and the 'vector constraint', (12), of [1] implies $E_2 = \pm E_3$. Thus $E_2f_3 - E_3f_2$ vanishes and so does the last term in (2).

generated by the constraints on certain types of initial data. Specifically, we analysed data characterized by N (an arbitrary positive integer) non simply connected annuli on the torus where the curvature was null and with zero curvature elsewhere (the torus is a product of 2 circles and can be coordinatized by 2 angles (θ, ϕ); each annulus lies in some interval of θ with ϕ going through its full range). We showed that the action of all the constraints restricted to the connection part of the N patch data is some combination of a 2d diffeomorphism and an SO(2,1) gauge rotation. Thus, flat and null patches can only be moved around but never created or destroyed by the action of the constraints. This meant that if we could actually build such solutions to the constraints in a way that holonomies of the connections around non-contractible loops in the flat sectors were freely specifiable, the dimensionality of the reduced phase space would be infinite. The rationale for building the particular example of *initial data* for N patches in which the holonomies around loops in flat patches were freely specifiable, was to show that these types of solutions really do exist. It is very important to emphasize that we never had to fix any gauge, (there is no BV gauge!) so it is impossible that we arrived at any "erroneous" conclusion as a consequence of the use of a partial gauge fixing procedure as Manojlović and Miković claim.

For the sake of completeness we repeat now the arguments of [2] explicitly in the context of [1]. To this end we consider initial data of the type [2],[1]:

$$\tilde{E}^\theta_I = E_1 x_I, \quad A^I_\theta = A_1 x^I, \quad \tilde{E}^\phi_I = E_2 y_I + E_3 t_I, \quad A^I_\phi = A_2 y^I + A_3 t^I \tag{6}$$

where E_1, E_2, E_3, A_1, A_2, A_3 are functions of θ (we coordinatize the torus with θ, $\phi \in [0, 2\pi)$ and identify 0 and 2π) and x_I, y_I, t_I are an orthonormal basis in the Lie algebra of $SO(2,1)$. This ansatz *is not a gauge fixing condition*; its purpose is only to select a subset of the possible initial data in order to study them in more detail. Inserting (6) in the gauge transformations generated by the 2+1 dimensional

Ashtekar constraints with gauge parameters of the type [2]

$$N^I = \epsilon^1(\theta) x^I, \quad N^a \partial_a = \epsilon^2(\theta)\partial_\theta, \quad \underset{\sim}{N} = \epsilon^3(\theta) \tag{7}$$

we find equations

$$\delta A_1 = -\frac{d\epsilon^1}{d\theta} + (E_2 f_3 - E_3 f_2)\epsilon^3 \tag{8}$$
$$\delta A_2 = -A_3 \epsilon^1 + f_2 \epsilon^2 + E_1 f_3 \epsilon^3 \tag{9}$$
$$\delta A_3 = -A_2 \epsilon^1 + f_3 \epsilon^2 + E_1 f_2 \epsilon^3 \tag{10}$$

Note that due to the error described in **(i)** above, (10) replaces the incorrect equation (19) of [1].

Let us consider the infinitesimal transformations on a null connection for which $f_2 = f_3$ and $E_2 = E_3$ (a similar analysis for $f_2 = -f_3$ is possible). Consider $\epsilon^1 = \epsilon^2 = 0$ i.e. a gauge transformation generated by the scalar constraint

$$\begin{aligned} \delta A_1 &= 0 \\ \delta A_2 &= E_1 f_3 \epsilon^3 \\ \delta A_3 &= E_1 f_2 \epsilon^3 \end{aligned} \tag{11}$$

If, instead, we consider the transformations with $\epsilon^1 = \epsilon^3 = 0$ (gauge transformations generated by the vector constraint equivalent to diffeomorphisms modulo $SO(2,1)$ transformations) we find

$$\begin{aligned} \delta A_1 &= 0 \\ \delta A_2 &= f_2 \epsilon^2 = f_3 \epsilon^2 \\ \delta A_3 &= f_3 \epsilon^2 = f_2 \epsilon^2 \end{aligned} \tag{12}$$

Thus, by choosing $\epsilon^2 = E_1 \epsilon^3$, the infinitesimal tranformations generated by the scalar constraint just reduce to diffeomorphisms (modulo $SO(2,1)$ transformations). For a flat connection this is true, as well, because we have now $\delta A_i = 0$, $i = 1, 2, 3$. As

[2] Notice that ϵ^1, ϵ^2, and, ϵ^3 correspond just to a subset of all the possible gauge transformations generated by the Gauss law, vector and scalar constraint respectively. In general, $N^I, N^a, \underset{\sim}{N}$ are functions of (θ, ϕ) with N^I, N^a pointing in any direction.

a consequence we conclude that the action of the scalar constraint smeared with a lapse ϵ^3 is equivalent to the action of the vector constraint with shift $\epsilon^2 = \epsilon^3 E_1$ as originally claimed in [2]. A direct consequence of this is that no gauge transformation can create or destroy patches because it is impossible to do this by the action of any diffeomorphism and/or $SO(2,1)$ gauge rotation.

(iv) Once the error in δA_3 is corrected, equation (46) in [1] gives

$$\delta W = -4\pi \frac{\sinh(\pi\sqrt{A_2^2 - A_3^2})}{\sqrt{A_2^2 - A_3^2}} \left[(A_2 f_2 - A_3 f_3)\epsilon^2 + E_1 (A_2 f_3 - A_3 f_2)\epsilon^3 \right]$$

which is zero inside flat patches, in perfect agreement with the results in [2].

In conclusion, we have shown that [1] has errors and does not point out any fallacy in our previous work [2]. There certainly exist interesting subtle open issues which impinge on the validity/relevance of [2] (whether our data are in a singular part of phase space (see [3, 4]), whether finite evolution could result in singular $\tilde{E}^a_I$ etc.), but these are not the issues discussed in [1].

References

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